QUANTUM OPTICS FOR BOSONS AND FERMIONS

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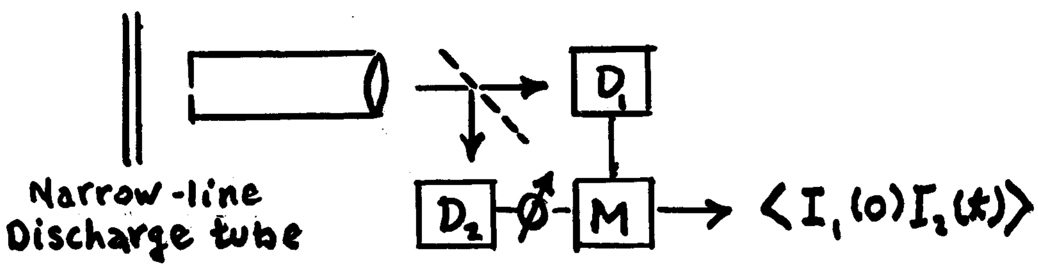
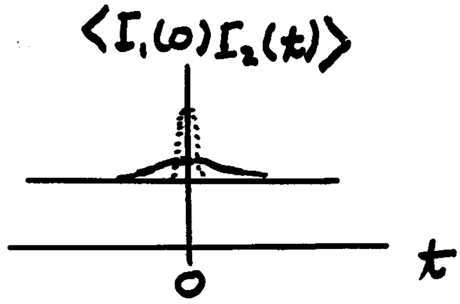
1956 - Hanbury Brown - Twiss Correlation Experiment



Observed:



Explanations:

Split quanta ?

HB-T

Purcell - Semiclassical, Assumed photodetection prob. $\sim \int_0^t E^2 \, dt$

Fluctuating E-field

Quantum field operator: $\quad E = E^{(+)} + E^{(-)}$

$$E^{(+)}(rt) = \sum_k \sqrt{\frac{\hbar\omega_k}{2}}\, \hat{e}(k)\, a_k\, e^{i(k\cdot r - \omega_k t)}$$

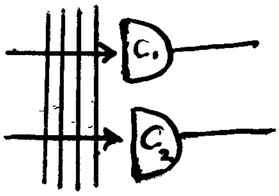
For ideal photon counter:

$$\text{Count rate} = w^{(1)}(rt) = s\langle E^{(-)}(rt)\, E^{(+)}(rt)\rangle$$
$$= s\,\mathrm{Tr}\{\rho\, E^{(-)}(rt)\, E^{(+)}(rt)\}$$
$$= s\, G^{(1)}(rt, rt),$$

where
the **First Order** correlation fn. is

$$G^{(1)}(rt\,r't') = \mathrm{Tr}\{\rho\, E^{(-)}(rt)\, E^{(+)}(r't')\}$$

For two counters –



Joint count rate (delayed coincidences)

$$w^{(2)}(rt, r't') = s^2 \, \mathrm{Tr}\{\rho E^{(-)}(rt) E^{(-)}(r't') E^{(+)}(r't') E^{(+)}(rt)\}$$

$$= s^2 G^{(2)}(rt \; r't', \; r't' \; rt)$$

~ with **Second Order** Correlation Fn:

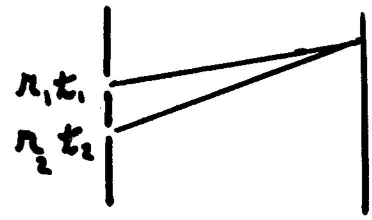
$$G^{(2)}(x_1 x_2 x_3 x_4) = \mathrm{Tr}\{\rho E^{(-)}(x_1) E^{(-)}(x_2) E^{(+)}(x_3) E^{(+)}(x_4)\}$$

$$x = (\vec{r}, t)$$

Note: Normal ordering of field operators

Define correlation fn. $G^{(1)}(r_1 t_1 r_2 t_2) = \langle E^{(-)}(r_1 t_1) E^{(+)}(r_2 t_2) \rangle$

Young's 2-pinhole exp't:



~ measures $G^{(1)}(r_1 t_1 r_1 t_1) + G^{(1)}(r_2 t_2 r_2 t_2) + 2 \mathrm{Re}\, G^{(1)}(r_1 t_1 r_2 t_2)$

coherence maximizes fringe contrast

Schwarz inequality: $|G^{(1)}(x_1 x_2)|^2 \leq G^{(1)}(x_1 x_1) G^{(1)}(x_2 x_2)$

Optical coherence: $|G^{(1)}(x_1 x_2)|^2 = G^{(1)}(x_1 x_1) G^{(1)}(x_2 x_2)$

Sufficient condition: $G^{(1)}$ factorizes

i.e. $G^{(1)}(x_1 x_2) = \mathcal{E}^*(x_1) \mathcal{E}(x_2)$

~ also necessary

Titulaer + G. Phys. Rev. 140 (1965)
145 (1966)

Define higher order coherence: e.g. Second order

$$G^{(2)}(x_1 x_2 x_3 x_4) = \langle E^{(-)}(x_1) E^{(-)}(x_2) E^{(+)}(x_3) E^{(+)}(x_4) \rangle$$
$$= \mathcal{E}^*(x_1) \mathcal{E}^*(x_2) \mathcal{E}(x_3) \mathcal{E}(x_4)$$

$\Longrightarrow$ Joint count rate factorizes

$$G^{(2)}(x_1 x_2 x_2 x_1) = |\mathcal{E}(x_1)|^2 |\mathcal{E}(x_2)|^2$$

$\Longrightarrow$ Wipes out HB-T correlation

n-th order coherence $n \to \infty$

What field states factorize all $G^{(n)}$?

Recall normal ordering

Sufficient to have $E^{(+)}(r t) | \rangle = \mathcal{E}(r t) | \rangle$

~ defines coherent states

Convenient basis for averaging normally ordered products

Many-mode fields: $\{a_k\} \to \{\alpha_k\}$ in coh. states

$$E^{(+)}(r\,t) = i \sum_k \sqrt{\frac{\hbar\omega_k}{2}}\, a_k\, u_k(r)\, e^{-i\omega_k t}$$

$$\mathcal{E}(r\,t) = i \sum_k \sqrt{\frac{\hbar\omega_k}{2}}\, \alpha_k\, u_k(r)\, e^{-i\omega_k t}$$

In a Coh. state of field: $E^{(+)}(r\,t)|\{\alpha_k\}\rangle = \mathcal{E}(r\,t)|\{\alpha_k\}\rangle$

$$G^{(1)}(r\,t\;r'\,t') \equiv \langle E^{(-)}(r\,t)\, E^{(+)}(r'\,t')\rangle$$
$$= \mathcal{E}^*(r\,t)\, \mathcal{E}(r'\,t')$$

~ First order coherence

Coherence $\longleftrightarrow$ Factorization

Let $x_i = (r_i\; t_i)$

Full coherence , $n = 1, 2, 3, \ldots$

$$G^{(n)}(x_1 \cdots x_{2n}) = \mathcal{E}^*(x_1) \cdots\cdots \mathcal{E}(x_{2n})$$

Quantum Field Theory – for Bosons

Field oscillation modes $\longleftrightarrow$ harmonic oscillators

For harmonic oscillator:

a lowers excitation $a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger$ raises excitation $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$a a^\dagger - a^\dagger a = 1$$

Special states: $a|\alpha\rangle = \alpha|\alpha\rangle$

α = any complex number

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$P(n) = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}$$, Poisson distrib.

$$\langle n\rangle = |\alpha|^2$$

~ Coherent States

Coherent states are displaced Oscillator ground states

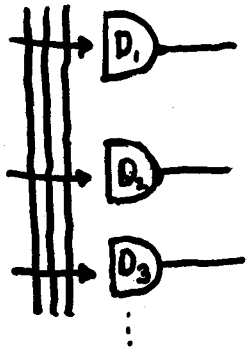
$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

$$D^{-1}(\alpha)\, a\, D(\alpha) = a + \alpha$$

$$|\alpha\rangle = D(\alpha)|0\rangle$$

Full coherence $\longrightarrow$ $P(n)$ = Poisson distrib.

n-detector joint counting rates



$$G^{(n)}(x_1 \cdots x_n\, x_n \cdots x_1) = \prod_{j=1}^{n} G^{(1)}(x_j\, x_j)$$

~ Uncorrelated   <u>No</u> HB-T effect

Ordinary - non coherent - light sources have randomly distributed amplitudes α_k.

D.L.

e.g. If the density operator for a single mode can be written as: $\rho = \int P(\alpha)\, |\alpha\rangle\langle\alpha|\, d^2\alpha$

then $\langle a^{\dagger n} a^m \rangle = \mathrm{Tr}(\rho\, a^{\dagger n} a^m) = \int P(\alpha)\, \alpha^{*n} \alpha^m\, d^2\alpha$

operator averages become integrals

$P(\alpha)$ = quasiprobability density

Scheme works well for pseudoclassical fields

For those electromagnetic superposition principle takes convolution form:

$$P(\alpha) = \int P_1(\alpha - \alpha')\, P_2(\alpha')\, d^2\alpha'$$

For other states $P(\alpha)$ may take on negative values - or become uselessly singular. Can we realize such states?

Yes!

For a Laser beam: $|\alpha\rangle$

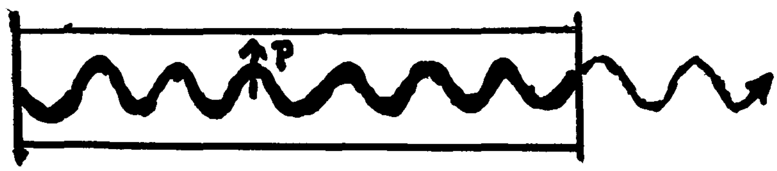
or, more generally, random phase

mixture $\rho = \frac{1}{2\pi}\int |Ae^{i\theta}\rangle\langle Ae^{i\theta}|\, d\theta$

(Radiating element is strong

– essentially classical – oscillating

electric polarization current $\frac{\partial P}{\partial t}$)



$\Longrightarrow$ Laser beam contains NO

HB-T Correlations

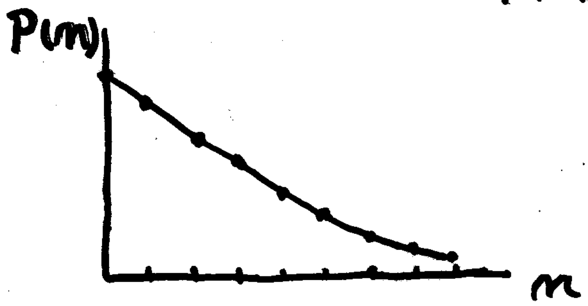
R.G. Phys. Rev. 130 '63

e.g. Gaussian distrib. of amplitudes $\{\alpha_k\}$

Single-mode density operator

$$\rho_{chaotic} = \frac{1}{\pi\langle n\rangle}\int e^{-\frac{|\alpha|^2}{\langle n\rangle}} |\alpha\rangle\langle\alpha|\, d^2\alpha$$

$$= \frac{1}{1+\langle n\rangle}\sum_{j=0}^{\infty}\left(\frac{\langle n\rangle}{1+\langle n\rangle}\right)^j |j\rangle\langle j|$$



Two-Fold joint count rate:

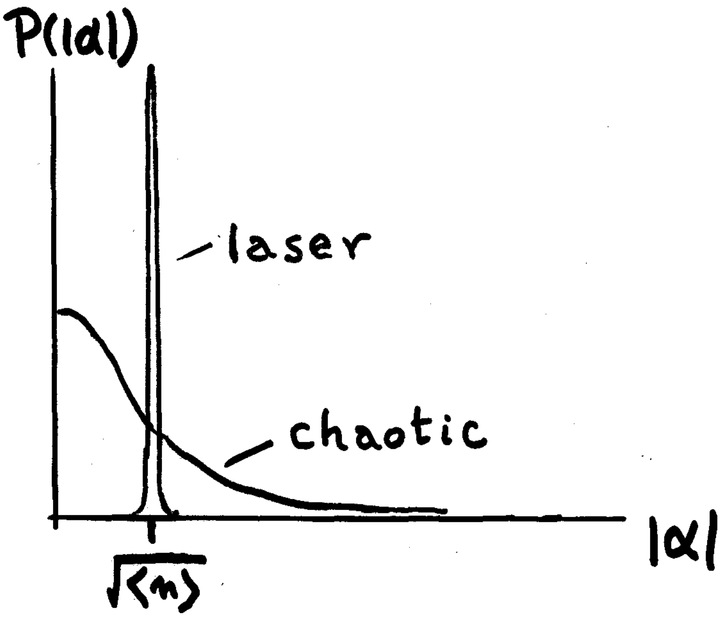
$$G^{(2)}(x_1 x_2\, x_2 x_1) = G^{(1)}(x_1 x_1)\, G^{(1)}(x_2 x_2) + \underbrace{G^{(1)}(x_1 x_2)\, G^{(1)}(x_2 x_1)}_{\text{HB-T Effect}}$$

Note for $x_2 \to x_1$

$$G^{(2)}(x_1 x_1\, x_1 x_1) = 2\left[G^{(1)}(x_1 x_1)\right]^2$$

D.L.

One mode excitation



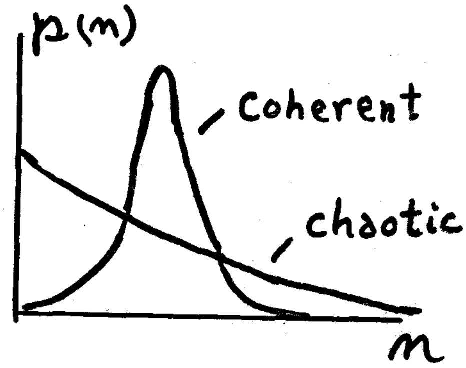
Photocount distrib's (w = average count rate)

Coherent state:

$$p(n) = \frac{(wt)^n}{n!} e^{-wt}$$

Chaotic state:

$$p(n) = \frac{(wt)^n}{(1+wt)^{n+1}}$$



Distrib. of time intervals till first count

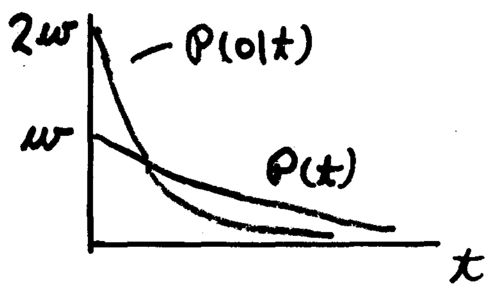
coherent: $\mathcal{P}(t) = w e^{-wt}$

chaotic: $\mathcal{P}(t) = \dfrac{w}{(1+wt)^2}$

Given count at $t=0$, distrib. of intervals till next count

Coherent: $\mathcal{P}(0|t) = w e^{-wt}$

chaotic: $\mathcal{P}(0|t) = \dfrac{2w}{(1+wt)^3}$



Quasiprobability representations for Quantum State ρ

Define characteristic functions:

$$\chi(\lambda, s) = \mathrm{Trace}\left\{\rho\, e^{\lambda a^{+} - \lambda^{*} a}\right\} e^{\frac{s}{2}|\lambda|^2}$$

Family of quasiprobability densities:

$$W(\alpha, s) = \frac{1}{\pi}\int e^{\alpha\lambda^{*} - \alpha^{*}\lambda}\, \chi(\lambda, s)\, d^2\lambda$$

$s = 1$	$W(\alpha, 1) = P(\alpha)$	P-Rep.
$s = 0$	$W(\alpha, 0) = W(\alpha)$	Wigner fn.
$s = -1$	$W(\alpha, -1) = \frac{1}{\pi}\langle\alpha\|\rho\|\alpha\rangle$	Q-Rep.

For fermion fields —

Anticommutation:

$$a_n a_m^\dagger + a_m^\dagger a_n \equiv \{a_n, a_m^\dagger\} = \delta_{nm}$$

$$\{a_n, a_m\} = 0 \qquad \{a_n^\dagger, a_m^\dagger\} = 0$$

Vac. State: $|0\rangle$, $a_n|0\rangle = 0$

Introduce Grassmann Variables: γ_n . γ_n^*

$$\{\gamma_n, \gamma_m\} = 0 \qquad \{\gamma_n, \gamma_m^*\} = 0 \qquad \{\gamma_n^*, \gamma_m^*\} = 0$$

~ Anticommute with a_n, $a_n^\dagger$

$$\{\gamma_n, a_m\} = 0 \qquad \text{for all } n, m$$

Note: $\gamma_n^2 = 0$, $\gamma_n^{*2} = 0$ etc.

Introduce Displacement Operator

$$D(\gamma) = e^{a^\dagger \gamma - \gamma^* a} = 1 + a^\dagger \gamma - \gamma^* a - \tfrac{1}{2}(a^\dagger a - a a^\dagger)\gamma\gamma^*$$

$$D^\dagger(\gamma)\, a\, D(\gamma) = a + \gamma$$

Multimode Displacement: $D(\{\gamma_k\}) = \prod_k e^{a_k^\dagger \gamma_k - \gamma_k^* a_k}$

Baker-Hausdorf Thm. $e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]}$

$$D(\gamma) = e^{a^\dagger \gamma} e^{-\gamma^* a} e^{\frac{1}{2}\gamma\gamma^*}$$

Coherent States:

$$|\gamma\rangle \equiv D(\gamma)|0\rangle = e^{\frac{1}{2}\gamma\gamma^*} e^{a^\dagger \gamma} |0\rangle$$

$$= (1 + \tfrac{1}{2}\gamma\gamma^*)(1 + a^\dagger \gamma)|0\rangle = (1 + \tfrac{1}{2}\gamma\gamma^* + a^\dagger \gamma)|0\rangle$$

$$a|\gamma\rangle = \gamma|\gamma\rangle$$

Cahill + R.G. PRA 59 1538 ('99)

Scalar Products:

$$\langle \gamma | \beta \rangle = e^{\gamma^*\beta - \frac{1}{2}(\gamma^*\gamma + \beta^*\beta)}$$

$$\langle \beta | \gamma \rangle \langle \gamma | \beta \rangle = e^{-(\beta^* - \gamma^*)(\beta - \gamma)}$$

vs. $|\langle \beta | \gamma \rangle|^2 = e^{-|\beta - \gamma|^2}$
for bosons

Eigenstates of Creation Operators:

$$|\alpha\rangle' \equiv D(\alpha)|1\rangle \quad , \quad |1\rangle = \text{filled state} = a^+|0\rangle$$

$$a^+|\alpha\rangle' = \alpha^*|\alpha\rangle'$$

Define <u>Left</u> differentiation

e.g. $\frac{d}{d\xi}(\mu + \xi t) = t$

Berezin integration rules:

$$\int d\alpha_n = 0 \qquad \int d\alpha_n^* = 0$$

$$\int d\alpha_n \, \alpha_m = \delta_{nm} \qquad \int d\alpha_n^* \, \alpha_m^* = \delta_{nm}$$

QOF

For Bosons <u>and</u> Fermions:

General quasiprob. distrib. $W(\alpha, s)$

$s = 1$ $\quad W(\alpha; 1) = P(\alpha)$

$$\rho = \int d^2\alpha \; P(\alpha) \, |\alpha\rangle\langle \pm\alpha|$$

P-representation

$s = 0$ $\quad W(\alpha, 0) = W(\alpha)$

Wigner representation

$s = -1$ $\quad W(\alpha, -1) = Q(\alpha)$

$$= \frac{1}{\pi} \langle \alpha | \rho | \pm\alpha \rangle$$

Q-representation

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Characteristic Functions:

$$\chi(\xi) = Tr\left(\rho\, e^{\xi a^{\dagger} - a\xi^{*}}\right)$$

$$\chi_N(\xi) = Tr\left(\rho\, e^{\xi a^{\dagger}}\, e^{-a\xi^{*}}\right) = \chi(\xi)\, e^{-\frac{1}{2}\xi\xi^{*}}$$

$\vdots$

$$\chi(\xi, s) = \chi(\xi)\, e^{-\frac{s}{2}\xi\xi^{*}}$$

$s=1$ normal
$s=0$ symm.
$s=-1$ antinormal

Define s-Ordered Quasiprob. Distrib.

$$W(\alpha, s) = \int d^2\xi\, e^{\alpha\xi^{*} - \xi\alpha^{*}}\, \chi(\xi, s)$$

e.g. For $s = 1$, Normal Ordering

Let $P(\alpha) = W(\alpha, 1)$

Then $$\rho = \int d^2\alpha \, P(\alpha) \, |\alpha\rangle\langle -\alpha|$$

is the P-representation of ρ, and

$$\mathrm{Tr}(\rho \, a^{+n} a^m) = \int d^2\alpha \, P(\alpha) \, \alpha^{*n} \alpha^m$$

Fermion Field: $\psi(x) = \sum_k a_k \phi_k(x)$

$x = (\vec{r}, t)$

Physical Correlation Function

$$G^{(m)}(x_1 \cdots x_n \, y_m \cdots y_1) = \mathrm{Tr}\{\rho \, \psi^+(x_1) \cdots \cdots \psi(y_1)\}$$

Define Grassmann Field

$$\varphi(x) = \sum_k \alpha_k \phi_k(x)$$

Field Coh. States obey

$$\psi(x) \, | \{\alpha_k\}\rangle = \varphi(x) \, | \{\alpha_k\}\rangle$$

Evaluate $G^{(m)}$ using P-representation

$$G^{(m)}(x_1 \cdots y_1) = \int \prod_k d^2\alpha_k \, P(\{\alpha_k\}) \, \varphi^*(x_1) \cdots \varphi(y_1)$$

DOF

Chaotic States:

$$\rho_k = (1 - \langle n_k \rangle)|0\rangle\langle 0| + \langle n_k \rangle |1\rangle\langle 1|$$

$$\rho_{\text{Field}} \doteq \prod_k \rho_k$$

For such states –

$$P_k(\alpha) = W_k(\alpha, 1) = -\langle n_k \rangle e^{-\frac{\alpha\alpha^*}{\langle n_k \rangle}}$$

vs.

$$P_k(\alpha) = \frac{1}{\pi \langle n_k \rangle} e^{-\frac{|\alpha|^2}{\langle n_k \rangle}}$$

– for bosons

~GRAND-Canonical Ensemble

- for a non-interacting system

$$H = \sum \epsilon_k a_k^\dagger a_k \qquad N = \sum a_k^\dagger a_k$$

at temp. T

$$\beta = \frac{1}{k_B T} \qquad \mu = \text{Chemical pot.}$$

$$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

Ξ = Grand Partition Function

Density operator:

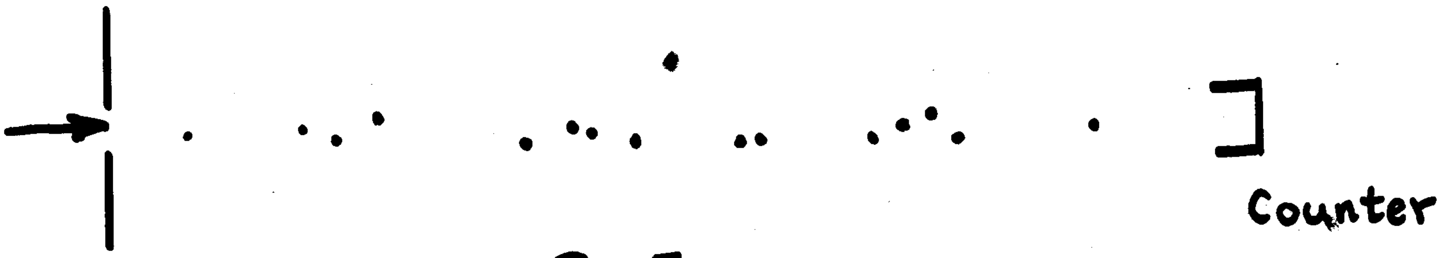
$$\rho = \frac{1}{\Xi(\beta,\mu)} e^{-\beta(H-\mu N)}$$

$$= \int \prod_k \left\{ \frac{1}{\pi\langle n_k \rangle} d^2\alpha_k \, e^{-\frac{\alpha_k \alpha_k^*}{\langle n_k \rangle}} |\alpha_k\rangle\langle\alpha_k| \right\}$$

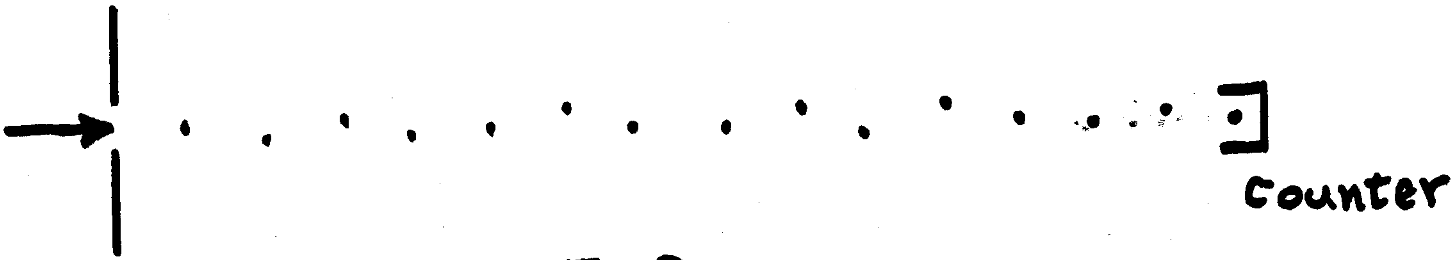
~an identity

QOF

Atom Counting



B.-E.



F.-D.

Measure $p(n,t)$

QOF

Model Beam: Rectangular spectrum

$$\gamma = \frac{\Delta\omega}{2\pi}$$

- State occupation numbers $\langle n \rangle < 1$

w = Average count rate

$$wt = K \int_0^t G^{(1)}(r\,t'\,r\,t')\, d\vec{r}\, dt$$

Beam Flux $= \langle n \rangle \gamma \geq w$

For 100% efficient counter:

$$\frac{w}{\gamma} = \langle n \rangle \leq 1$$

~ for Fermions

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Then for $\gamma t \gg 1$

$$Q_F(\lambda t) = \left(1 + \lambda \frac{w}{\gamma}\right)^{\gamma t}$$

$$Q_B(\lambda t) = \frac{1}{\left(1 - \lambda \frac{w}{\gamma}\right)^{\gamma t}}$$

$$p_F(n,t) = \frac{(-1)^n}{n!}\left(\frac{d}{d\lambda}\right)^n Q_F(\lambda,t)\Big|_{\lambda=1}$$

$$= \frac{\Gamma(\gamma t+1)}{n!\,\Gamma(\gamma t+1-n)}\left(\frac{w}{\gamma}\right)^n\left(1-\frac{w}{\gamma}\right)^{\gamma t-n}$$

$$p_B(n,t) = \frac{\Gamma(\gamma t+n)}{n!\,\Gamma(\gamma t)}\;\frac{(w/\gamma)^n}{\left(1+\frac{w}{\gamma}\right)^{\gamma t+n}}$$

Variances:

$$\langle c^2\rangle - \langle c\rangle^2 = \langle c\rangle\left(1 \mp \frac{w}{\gamma}\right)$$

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