Bose-Einstein correlations for

second order QCD phase transitions & the anomalous dimension of QCD

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- Bose-Einstein correlations in plane wave approximation:
  - Central limit theorem (CLT): Gaussian sources
  - Generalized CLT: Lévy stable laws
- Two and three particle correlations
- Measuring fractal structure of QCD jets – anomalous dimension
- At second order phase transitions: measuring critical exponent

Inspired by A. Bialas,
Acta Phys. Pol B23, 561 (1992)
(Brax and Peschanski, PLB 1991)

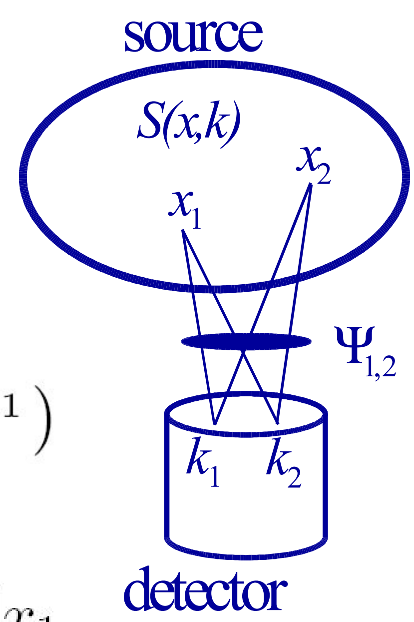
T. Cs, S. Hegyi, W. A. Zajc, nucl-th/0310042, Eur. Phys. J. C (2004)
T. Cs, S. Hegyi, W. A. Zajc, nucl-th/0402035, Nukleonika (2005)
T. Cs, T. Novák, S. Hegyi, W.A. Zajc, Acta Phys. Polonica B (2005)

Bose-Einstein correlations

- **Two plane-waves:**

$$\Psi_1 = \mathrm{e}^{-ik_1x_1}$$

$$\Psi_2 = \mathrm{e}^{-ik_2x_2}$$



- **Bosons: need for symmetrization**

$$\Psi_{1,2} = \frac{1}{\sqrt{2}}(\mathrm{e}^{-ik_1x_1}\mathrm{e}^{-ik_2x_2} + \mathrm{e}^{-ik_1x_2}\mathrm{e}^{-ik_2x_1})$$

Spectrum:

$$N_1(k_1) = \int S(x_1, k_1)|\Psi_1|^2\mathrm{d}x_1$$

- **Two-particle spectrum (momentum-distribution):**

$$N_2(k_1, k_2) = \int S(x_1, k_1)S(x_2, k_2)|\Psi_{1,2}|^2\mathrm{d}x_1\mathrm{d}x_2$$

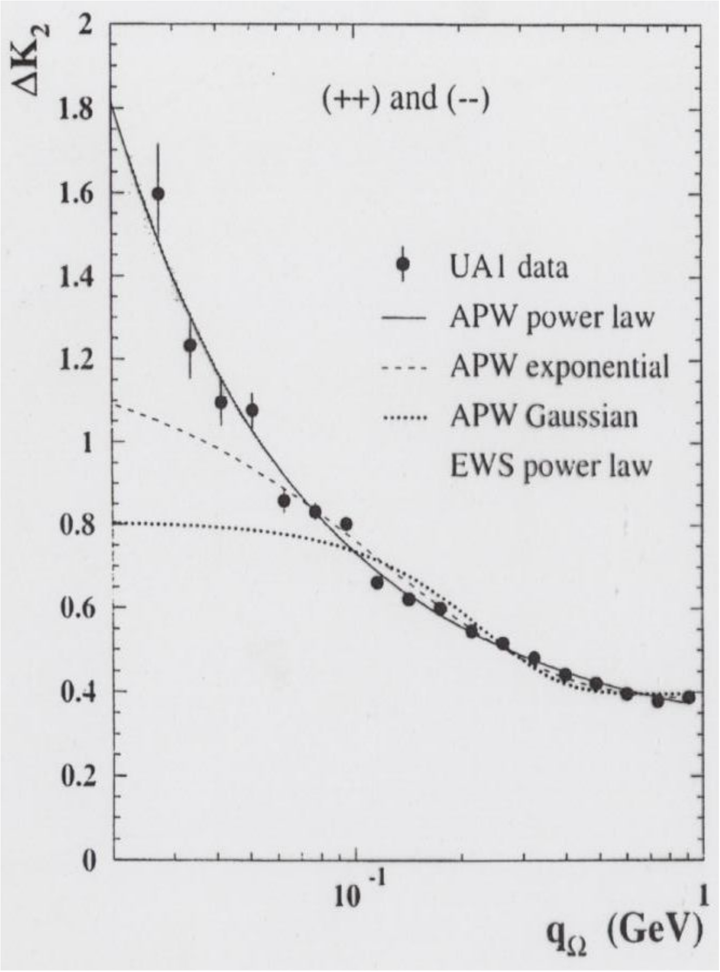
Approximations: Plane-wave, no multiparticle symmetrization, thermalization …

Non-Gaussian distributions

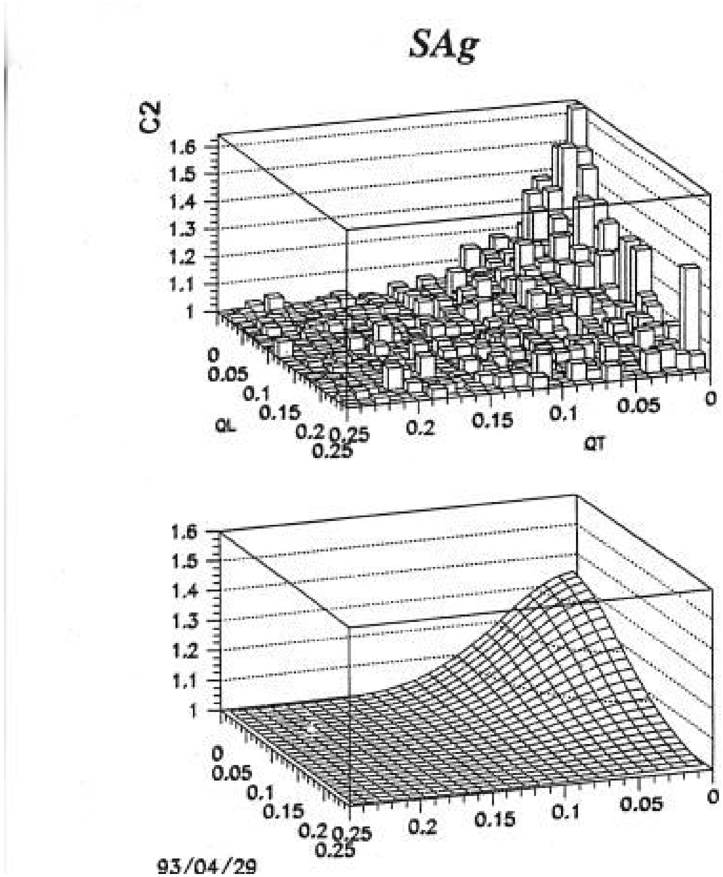
- **Of course, the source does NOT have to be Gaussian**
  - Non-Gaussian tails
  - Low-q bins

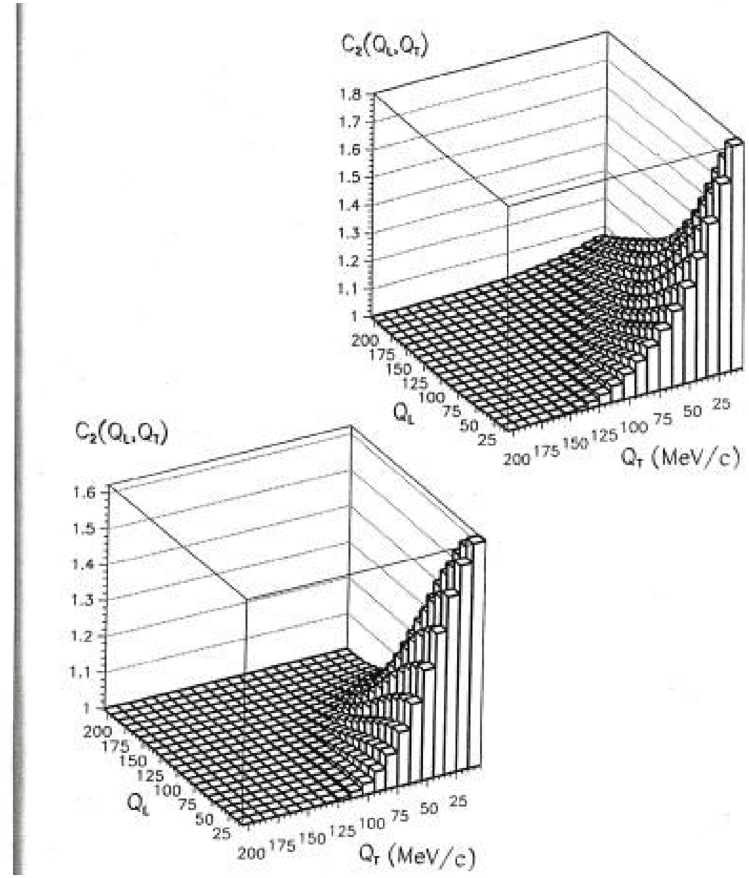
- **One can check, if the correlation function is really Gaussian or not**

- **The Gaussian assumption can potentially cause results to be meaningless**



2d examples of non-Gaussian correlations





Example of NA35 S+Ag 2d correlation data, and a Gaussian fit to it (lhs) which misses the peak around q=0. The rhs shows a 2d Edgeworth expansion fit to E802 Si+Au data at AGS (upper panel) compared to a 2d Gaussian fit for the same data set (lower panel).
Note the difference in the vertical scales.

Selfsimilarity, Lévy stable laws

$$x = \sum_n x_n$$

Hence the distribution of the sum x is obtained as an n-fold convolution,

$$f(x) = \int \mathrm{d}x_1 ... \mathrm{d}x_n f_1(x_1) ... f_n(x_n) \delta(x - x_1 - x_2 .... - x_n)$$

$$\tilde{f}(q) = \prod_{i=1}^{n} \tilde{f}_i(q)$$

$$\tilde{f}(q) = \exp\left(iq\delta - |\gamma q|^\alpha\right),$$

$$\tilde{f}_i(q) = \exp\left(iq\delta_i - |\gamma_i q|^\alpha\right), \qquad \prod_{i=1}^{n} \tilde{f}_i(q) = \exp\left(iq\delta - |\gamma q|^\alpha\right),$$

$$\gamma^\alpha = \sum_{i=1}^{n} \gamma_i^\alpha, \qquad \delta = \sum_{i=1}^{n} \delta_i,$$

Bose-Einstein C: Plane wave approximation

$$C_2(\mathbf{k}_1,\mathbf{k}_2) = \frac{N_2(\mathbf{k}_1,\mathbf{k}_2)}{N_1(\mathbf{k}_1)\, N_1(\mathbf{k}_2)}$$

experimental conditions:

i) The correlation function tends to a constant for large values of the relative momentum $q = k_1 - k_2$.

ii) Near $|q| = 0$, *the correlation function deviates from its asymptotic, large* $|q|$ *value in a certain domain of its argument.*

iii) The two-particle correlation function is related to a Fourier transformed space-time distribution of the source.

$$S(x,k) = f(x)\, g(k), \qquad \int \mathrm{d}x\, f(x) = 1, \qquad \int \mathrm{d}k\, g(k) = \langle n \rangle,$$

$$N_1(k) = \int \mathrm{d}x\, S(x,k) = g(k).$$

$$N_2(k_1,k_2) = \int \mathrm{d}x_1 \mathrm{d}x_2\, S(x_1,k_1)\, S(x_2,k_2)\, |\psi_{k_1,k_2}(x_1,x_2)|^2.$$

$$\psi_{k_1,k_2}(x_1,x_2) = \frac{1}{\sqrt{2}} \left[\exp(ik_1x_1 + ik_2x_2) + \exp(ik_1x_2 + ik_2x_1)\right].$$

$$C_2(k_1,k_2) = 1 + |\tilde{f}(q)|^2,$$

$$\tilde{f}(q) = \int \mathrm{d}x\, \exp(iqx)\, f(x), \qquad q = k_1 - k_2.$$

Extra Assumption: ANALYTICITY

$$C_2(k_1,k_2) = 1 + |\tilde{f}(q_{12})|^2,$$

$$\tilde{f}(q_{12}) = \int \mathrm{d}x \, \exp(iq_{12}x)\, f(x), \qquad q_{12} = k_1 - k_2.$$

$$\tilde{f}(q) \approx 1 + iq\langle x\rangle - q^2\langle x^2\rangle/2 + \ldots\,,$$

$$C(q) = 1 + |\tilde{f}(q)|^2 \approx 2 - q^2(\langle x^2\rangle - \langle x\rangle^2) \approx 1 + \exp(-q^2R^2),$$

$$R = \sqrt{\langle x^2\rangle - \langle x\rangle^2}.$$

Limit distributions, Lévy laws

The characteristic function for limit distributions is known also in the case, when the elementary process has infinite mean or infinite variance.
The simplest case, for symmetric distributions is:

Exact result
$$\tilde{f}(q) = \exp\left(iq\delta - |\gamma q|^{\alpha}\right),$$

Small q expansion
$$\tilde{f}(q) \approx 1 + iqx_0 - \frac{1}{2}|qR|^{\alpha}$$

This is not analytic function. The only case, when it is analytic, corresponds to the $\alpha = 2$ case.
The general form is

$$C(q;\alpha) = 1 + \exp\left(-|qR|^{\alpha}\right).$$

EXACT result

where $0 < \alpha <= 2$ is the Lévy index of stability.
4 parameters: center x_0,
scale R,
index of stability α
asymmetry parameter β

Examples in 1d

Cauchy or Lorentzian distribution, $\alpha = 1$

$$f(x) = \frac{1}{\pi} \frac{R}{R^2 + (x - x_0)^2}, \qquad -\infty < x < \infty,$$

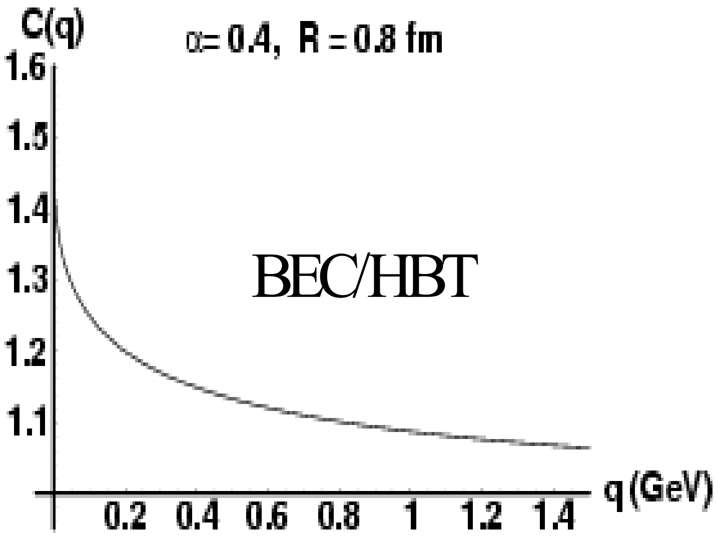
$$C(q) = 1 + \exp\left(-|q\,R|\right).$$

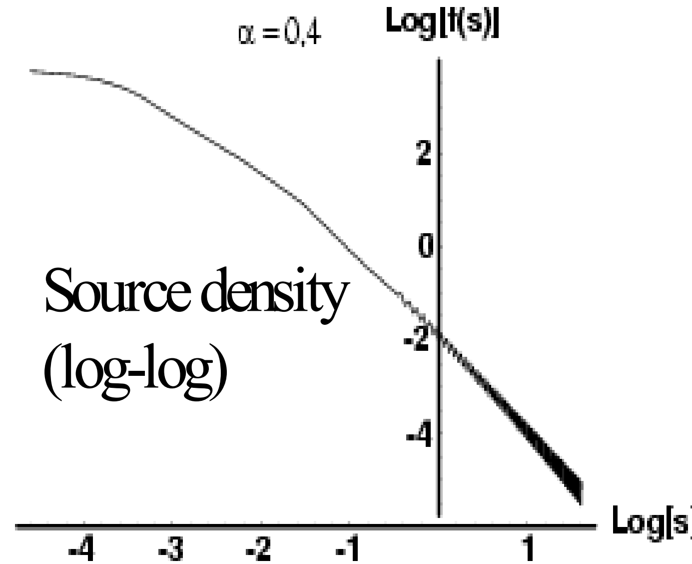
Asymmetric Levy distribution, has a finite, one sided support, $\alpha = 1/2$, $\beta = 1$

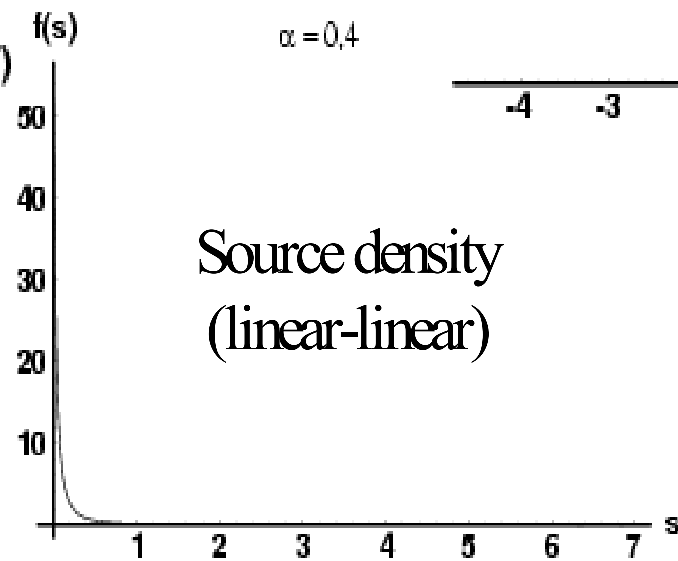
$$f(x) = \sqrt{\frac{R}{8\pi}} \frac{1}{(x - x_0)^{3/2}} \exp\left(-\frac{R}{8(x - x_0)}\right), \qquad x_0 < x$$

$$C(q) = 1 + \exp\left(-\sqrt{|q\,R|}\right).$$

Lévy sources and BE/HBT correlations



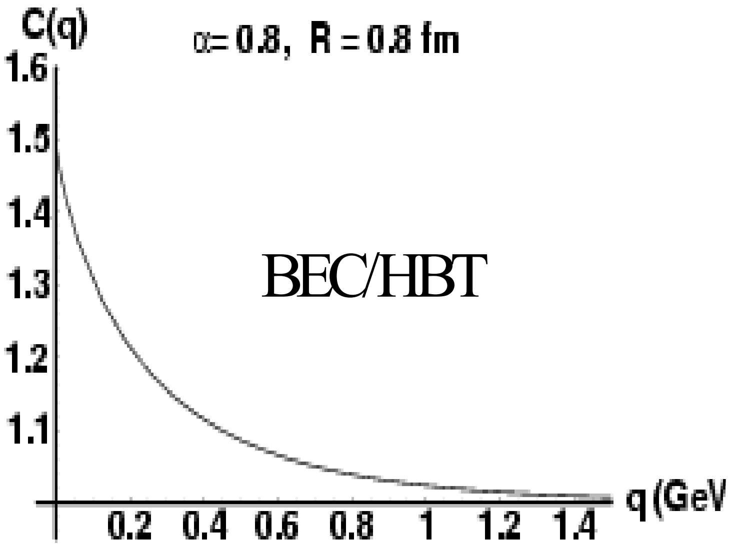


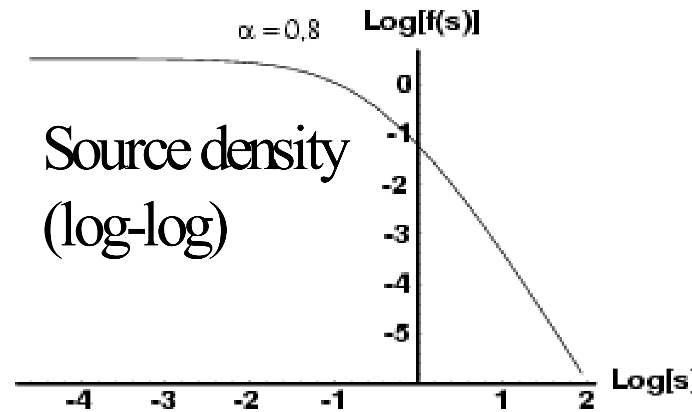


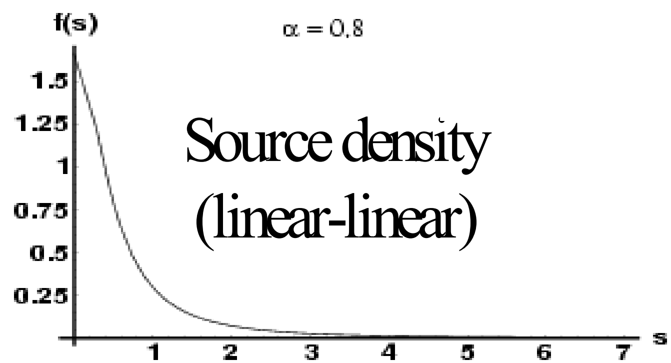
s=r/R
scaled coordinate

α **= 0.4: very peaked correlation function,**
strongly decreasing source density with power-law tail

Lévy sources and BE/HBT correlations



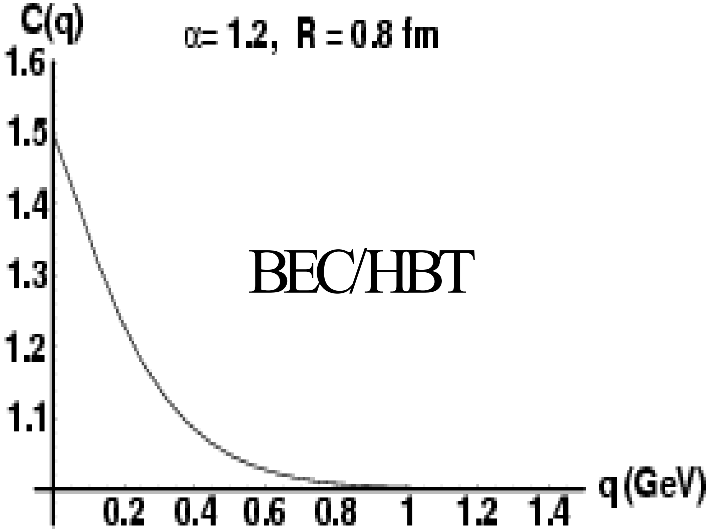


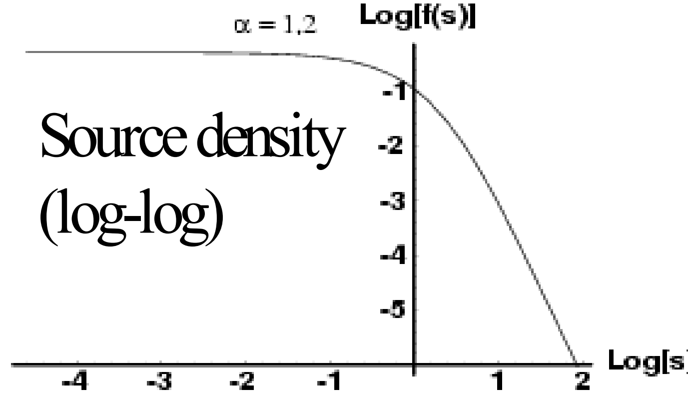


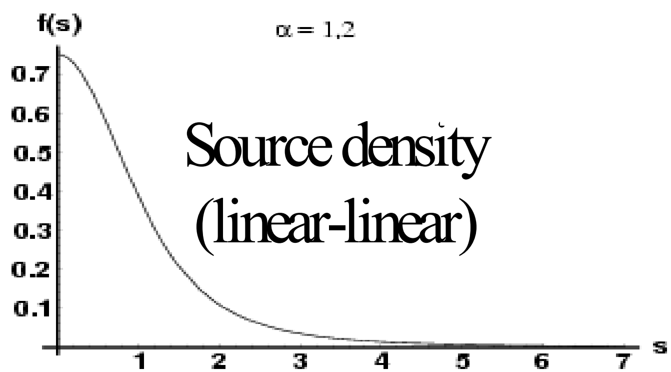
s=r/R
scaled coordinate

α = 0.8: peaked correlation function,
decreasing source density with power-law tail

Lévy sources and BE/HBT correlations



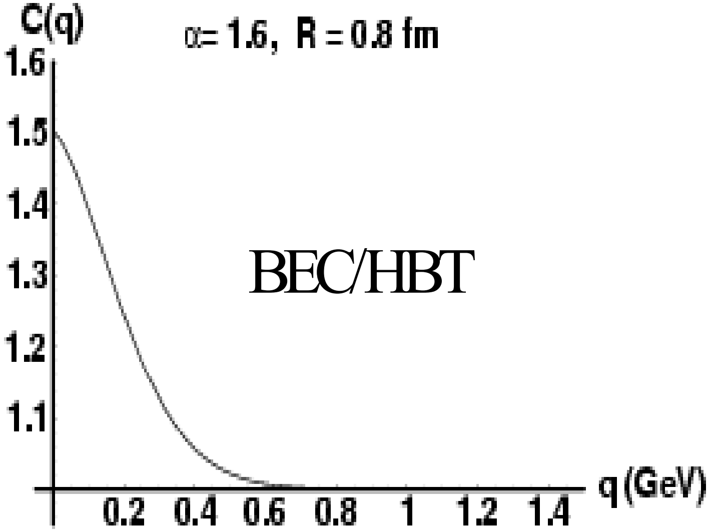


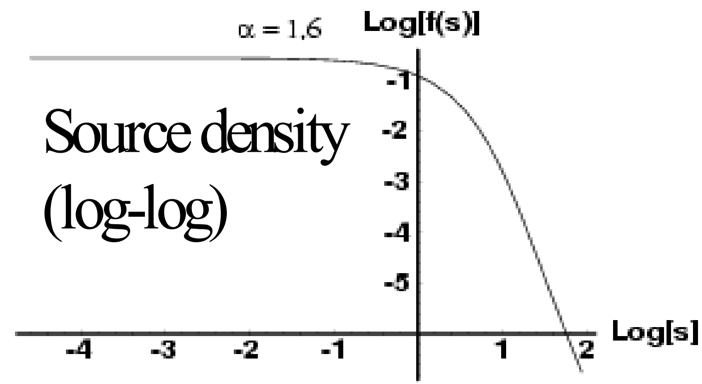


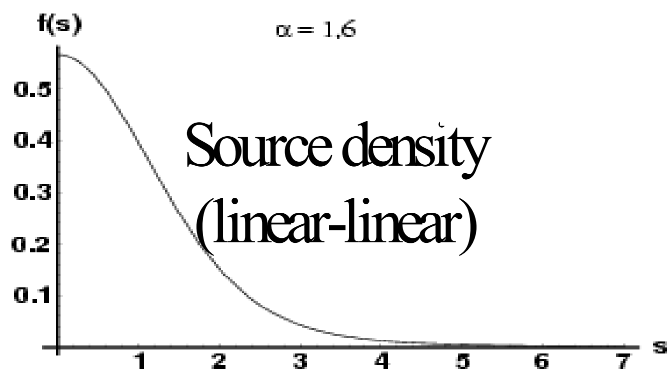
s=r/R
scaled coordinate

α = 1.2: less peaked correlation function,
less decreasing source density with power-law tail

Lévy sources and BE/HBT correlations



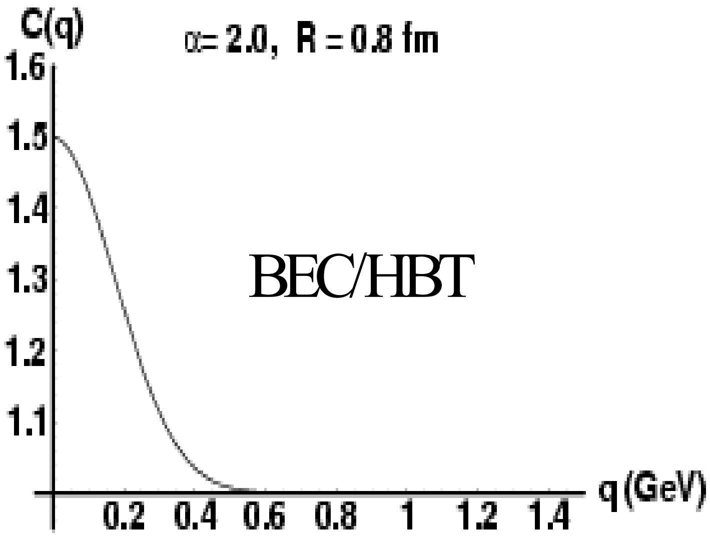


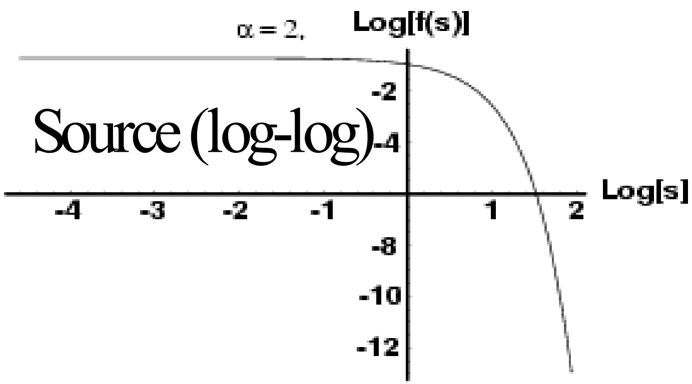


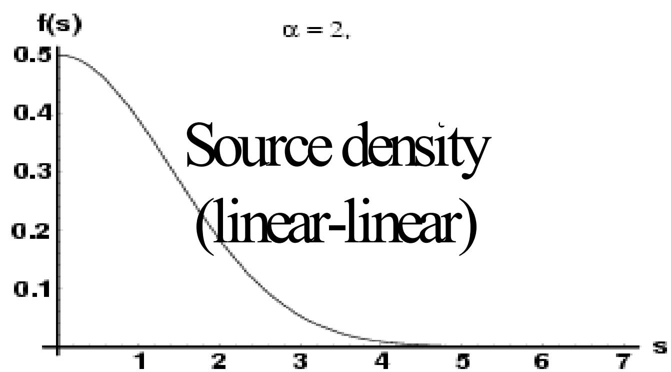
s=r/R
scaled coordinate

α = 1.6: correlation function can be mistaken as Gaussian source by eye looks like a Gaussian on lin-lin scale, but has a power-law tail on the log-log scale

Lévy sources and BE/HBT correlations







s=r/R
scaled coordinate

α = 2.0: really Gaussian correlation function,
really Gaussian source density,
power-law tail is gone

3d generalization

The case of symmetric Levy distributions is solved by

$$\tilde{f}(\mathbf{q}) = \exp\left(i\mathbf{q}\mathbf{x_0} - \frac{1}{2}\left|\sum_{i,j=1}^{3} R^2_{i,j} q_i q_j\right|^{\frac{\alpha}{2}} \right)$$

Multivariate symmetic Lévy Bose-Einstein correlations

$$C(\mathbf{q}) = 1 + \exp\left(-\left|\sum_{i,j=1,3} R^2_{i,j} q_i q_j\right|^{\frac{\alpha}{2}} \right)$$

the corresponding space-time distribution is given by in terms of R^{-1}, the inverse of the radius matrix

$$f_\alpha(s(\mathbf{x})) = \frac{1}{(2\pi \det R^2)^{d/2}} \int_0^\infty \mathrm{d}t\, t^{d-1}\, (ts(\mathbf{x}))^{1-d/2}\, J_{d/2-1}(ts(\mathbf{x})) e^{-t^\alpha},$$

$$s(\mathbf{x}) = |R^{-1}\mathbf{x}|$$

$$s(\mathbf{x}) = |R^{-1}\mathbf{x}| = \sqrt{\frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2}}.$$

Generalized 3d generalization

The case of symmetric Levy distributions for factorized time dependence

$$C_2(\mathbf{k}_1,\mathbf{k}_2) = 1 + \lambda \exp\left[-\Big(\sum_{i,j=s,o,l} R_{ij}^2 q_i q_j\Big)^{\frac{\alpha_x}{2}} - (\Delta t^2 \beta_o^2 q_o^2)^{\frac{\alpha_t}{2}}\right]$$

Correlation function: invariant
Q decomposition: invariant Buda-Lund variables
the Lévy version of invariant Buda-Lund variables

$$C(k_1,k_2) = 1 + \lambda \exp\left(-|R_= Q_=|^{\alpha_=} - |R_\parallel Q_\parallel|^{\alpha_\parallel} - |R_\perp Q_\perp|^{\alpha_\perp}\right),$$

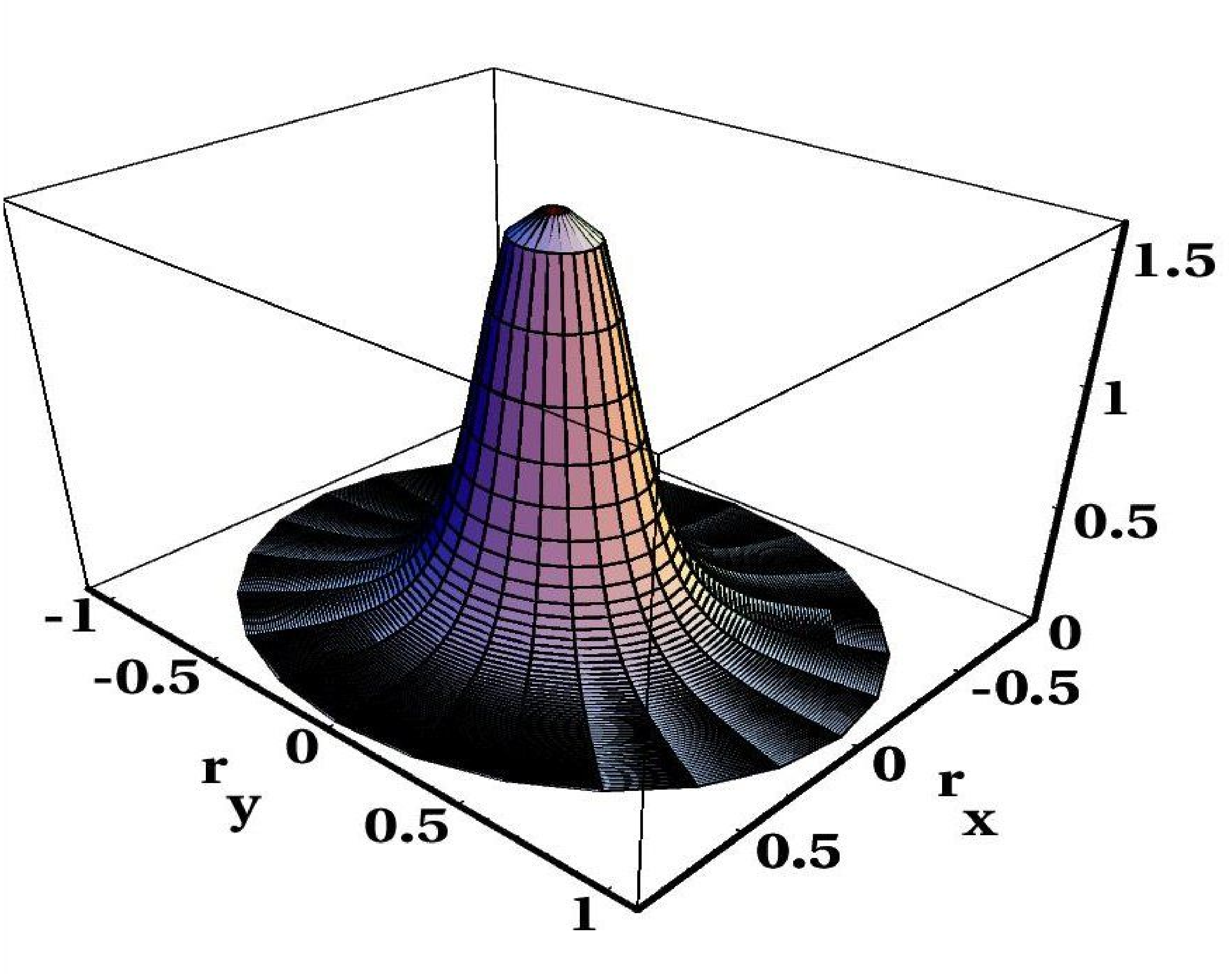
$$Q_= = m_{t,1}\cosh(y_1 - \overline{\eta}) - m_{t,2}\cosh(y_2 - \overline{\eta})$$

$$Q_\parallel = m_{t,1}\sinh(y_1 - \overline{\eta}) - m_{t,2}\sinh(y_2 - \overline{\eta})$$

$$Q_\perp = \sqrt{Q_x^2 + Q_y^2}.$$

2d Lévy sources, linear scale

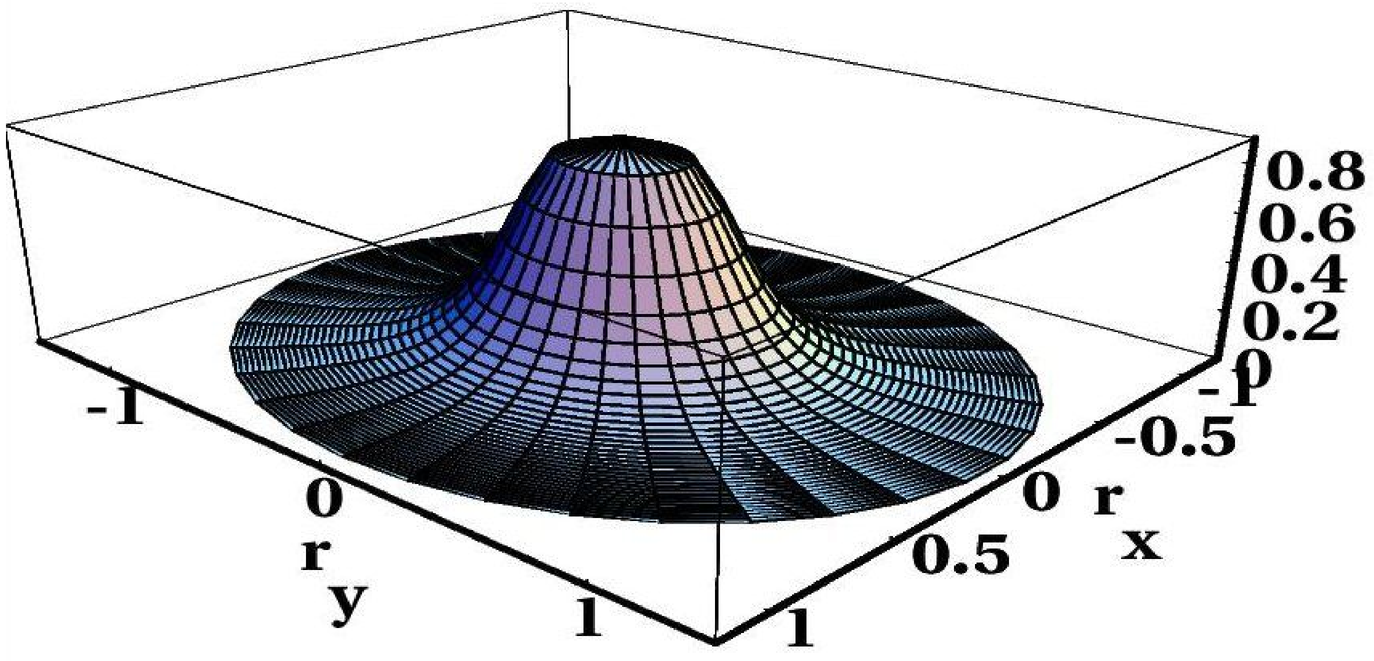
$\alpha = 0.6$



$s = r_x^{\,2}/X^2 + r_y^{\,2}/Y^2$ scaled coordinate: 1d problem for symmetric Lévy sources

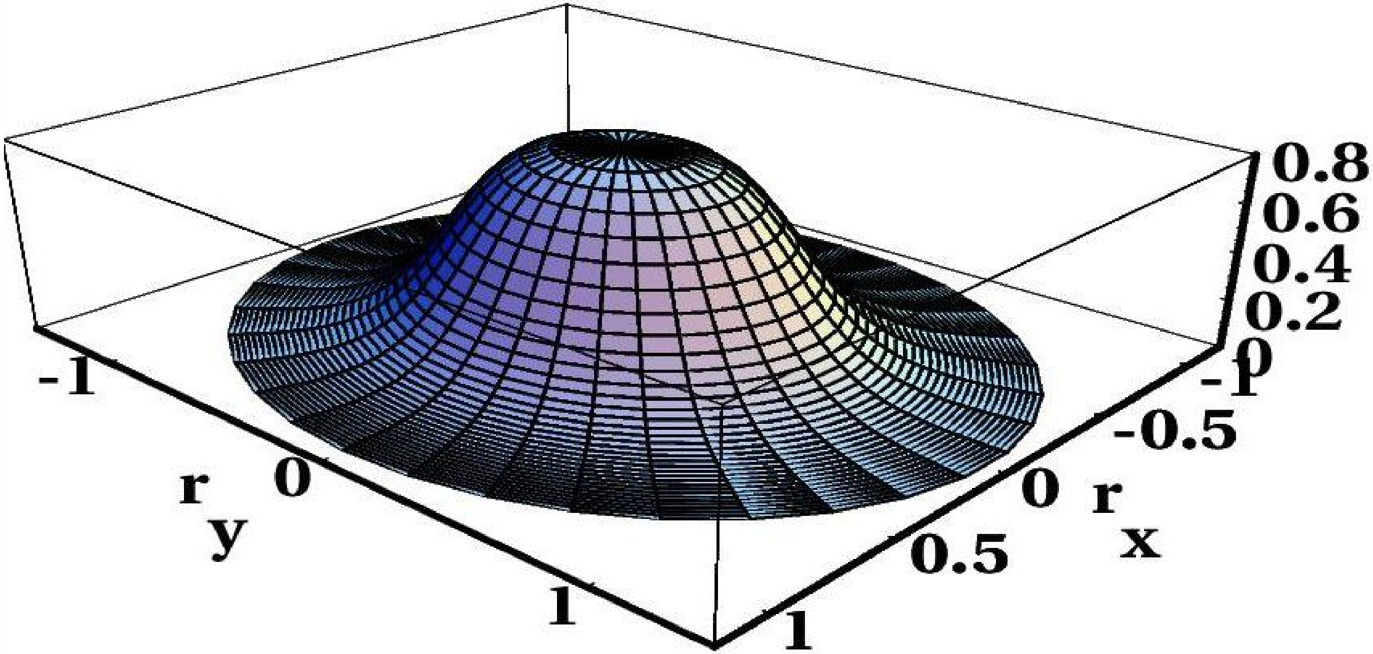
2d Lévy sources, linear scale

$\alpha = 0.8$



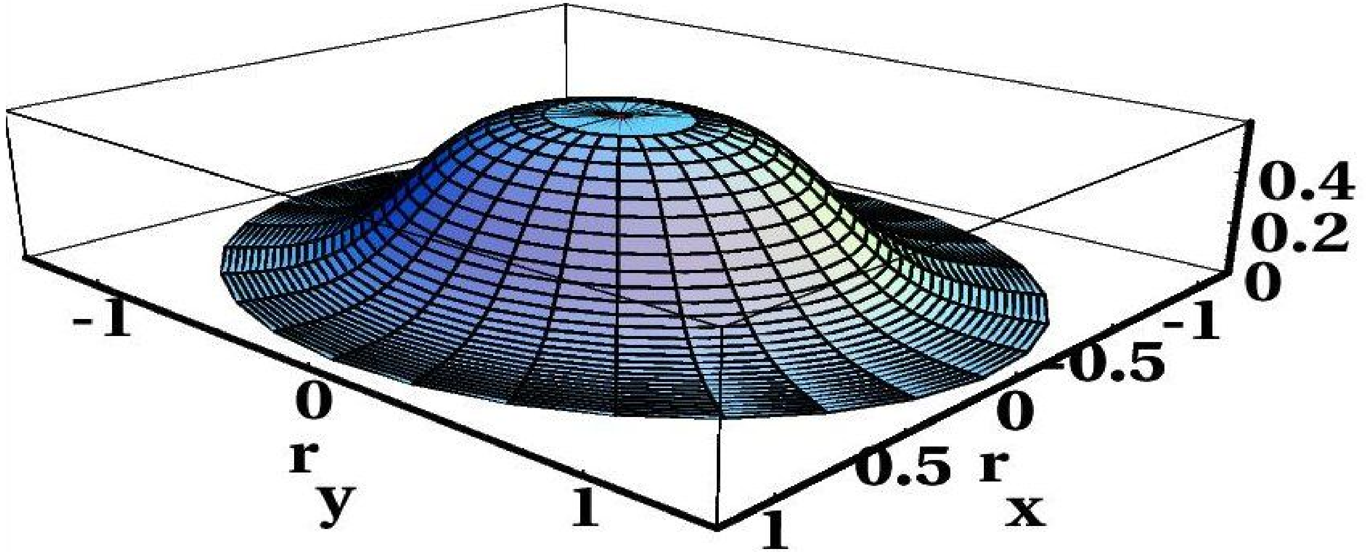
2d Lévy sources, linear scale

$\alpha = 1.2$



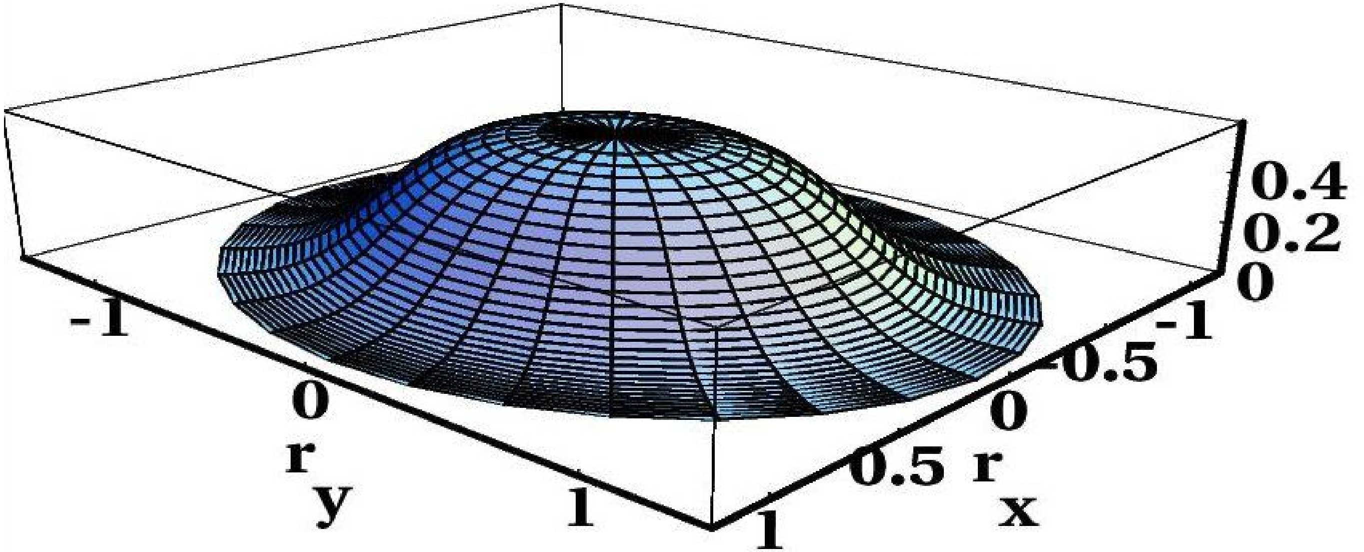
2d Lévy sources, linear scale

$\alpha = 1.6$



2d Lévy sources, linear scale

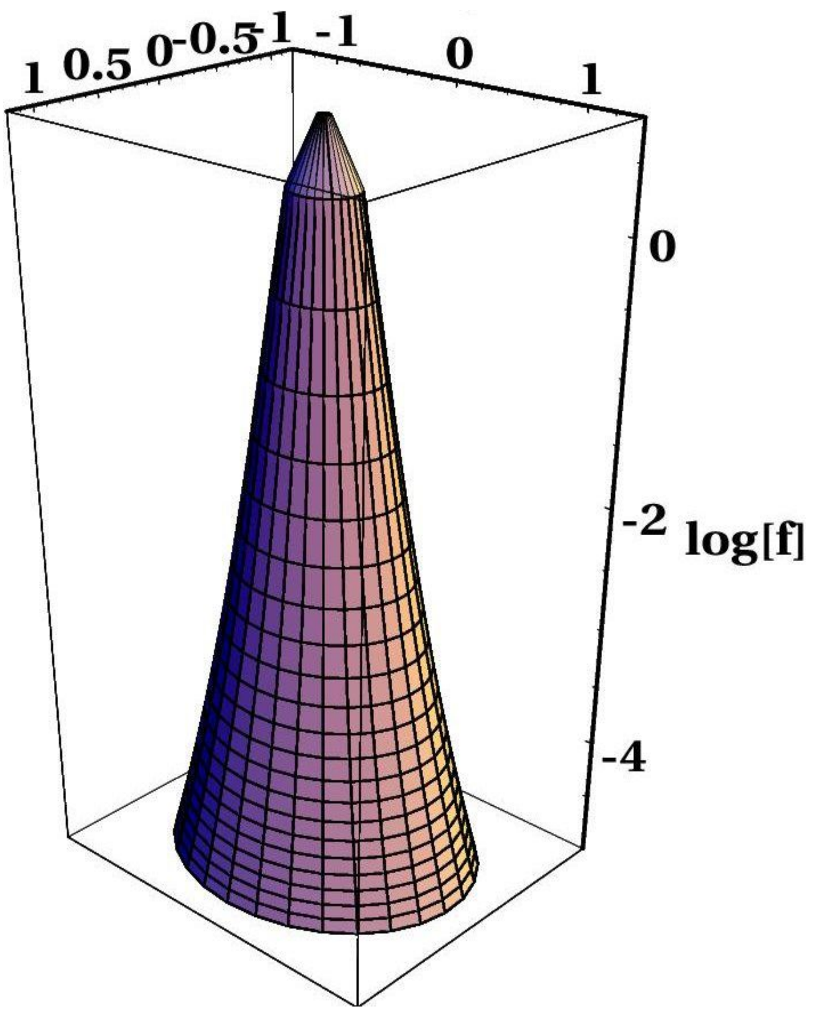
$\alpha = 2.0$



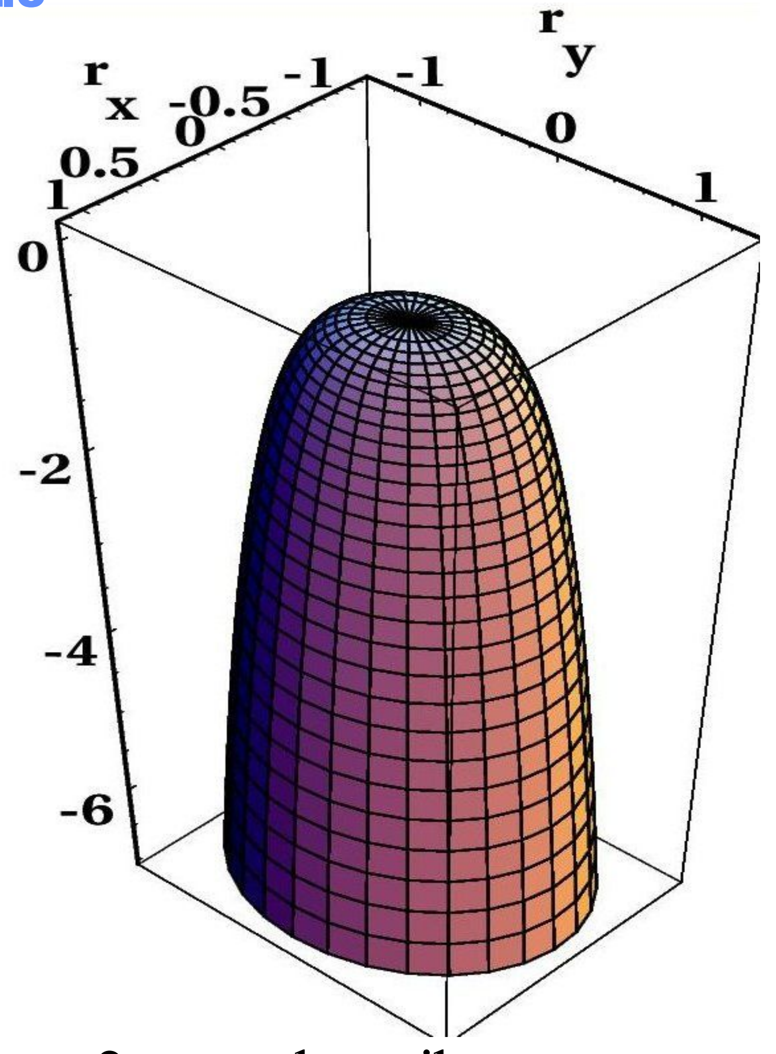
Finally, a Gaussian case!

2d Lévy sources, log -linear scale

$\alpha = 0.6$



$\alpha = 2.0$



For $\alpha < 2$: power-law tail

Asymmetric Lévy & 3-particle correlations

Normalized three-particle cumulant correlation function

$$w(1,2,3) = \frac{\kappa_3(1,2,3)}{2\sqrt{\kappa_2(1,2)\kappa_2(2,3)\kappa_2(3,1)}}$$

$$w(1,2,3) = \cos\left\{\frac{\beta}{2}R^\alpha \tan(\frac{\alpha\pi}{2})[\sum_{(i,j)})|q_{ij}|^\alpha \mathrm{sign}(q_{ij})]\right\}$$

$$w = \cos(\phi).$$

The angle ϕ is directly proportional to β, the asymmetry

(Multi)fractal jets in QCD

B. Andersson, P. Dahlquist, G. Gustafson, Nucl. Phys. B328 (1989) 76
G. Gustafson, A. Nilsson, Nucl. Phys. B355 (1991) 106, Z. Phys. C52 (1991) 533
G. Gustafson, Nucl. Phys. B392 (1993) 251
J. Samueslon, Yu. Dokshitzer, W. Ochs, G. Wilk ...

$$dP = \frac{C\alpha_S}{2\pi} dx_1 dx_3 \frac{(x_1^2 + x_3^2)}{(1-x_1)(1-x_3)}$$

$$x_2 + x_1 + x_3 = 2$$

$$k_\perp^2 \equiv W^2(1-x_1)(1-x_3)$$
$$y \equiv \frac{1}{2}\log\left(\frac{1-x_1}{1-x_3}\right)$$

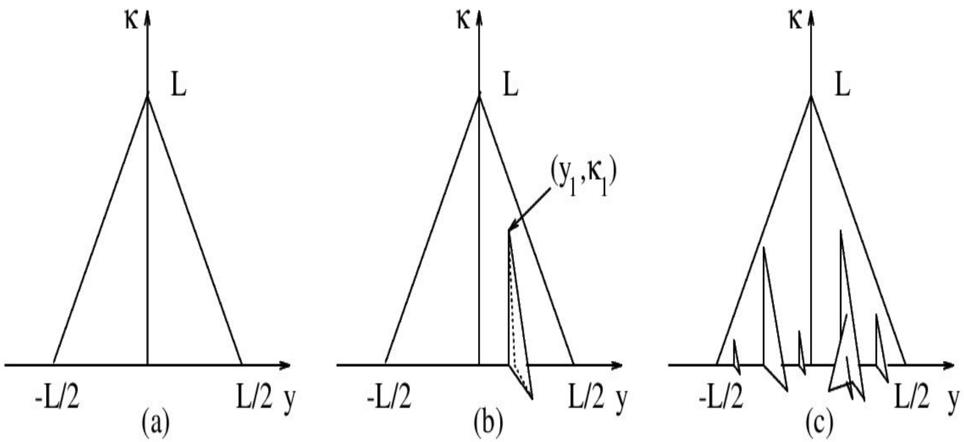


Figure 1: *(a) The phase space available for a gluon emitted by a high energy $q\bar{q}$ system is a triangular region in the y-κ plane. (b) If one gluon is emitted at (y_1, κ_1) the phase space for a second (softer) gluon is represented by the area of this folded surface. (c) Each emitted gluon increases the phase space for the softer gluons. The total gluonic phase space can be described by this multifaceted surface.*

$$dP \simeq \frac{C\alpha_s}{\pi}\frac{dk_\perp^2}{k_\perp^2}dy$$

$$k_\pm \equiv k_\perp \exp(\pm y) < W$$

a triangular region in the $(y, \kappa \equiv \log(k_\perp^2))$-plane

(Multi)fractal jets in QCD

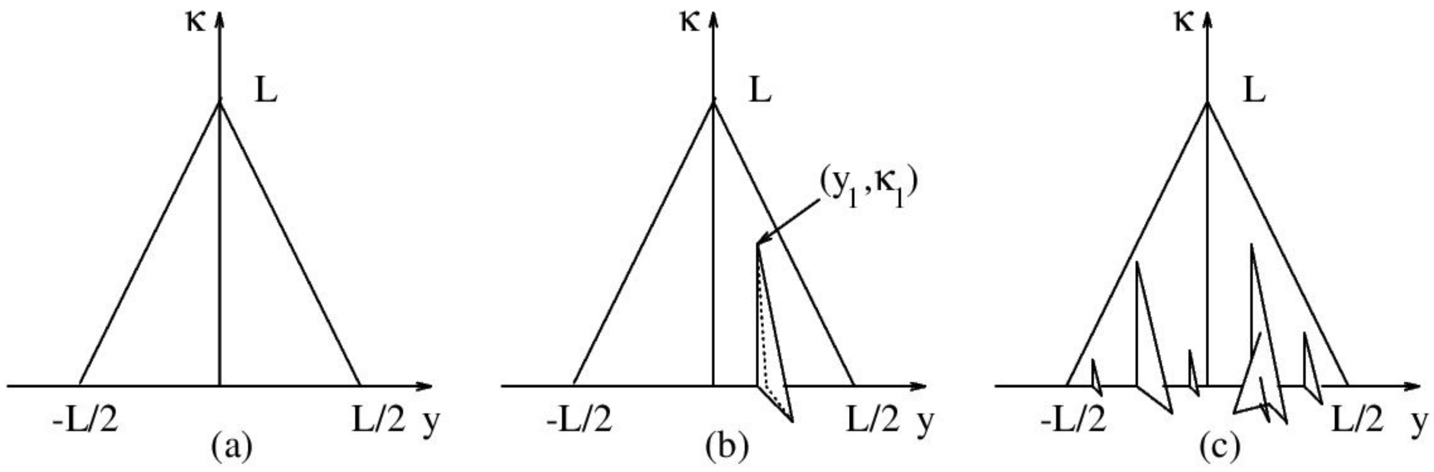


Figure 1: *(a) The phase space available for a gluon emitted by a high energy $q\bar{q}$ system is a triangular region in the y-κ plane. (b) If one gluon is emitted at (y_1, κ_1) the phase space for a second (softer) gluon is represented by the area of this folded surface. (c) Each emitted gluon increases the phase space for the softer gluons. The total gluonic phase space can be described by this multifaceted surface.*

The baseline forms a (multi)fractal, and the fractal dimension is given by the anomalous dimension of QCD: $1 + (3\,\alpha_s / 2\,\pi)^{0.5}$

Lund string, with string tension of 1 GeV/fm: maps the fractal in momentum space to coordinate space, without changing the fractal dimension.

Fits to NA22 and UA1 Bose-Einstein (BEC) data

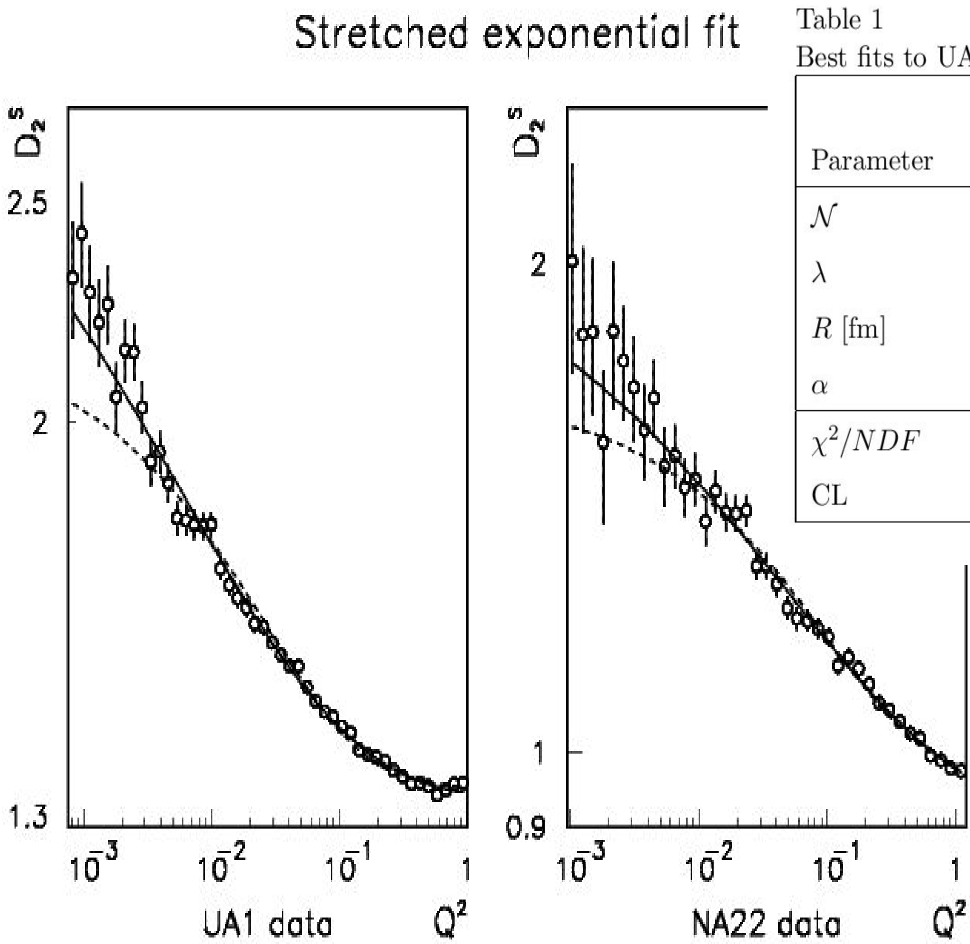


Table 1
Best fits to UA1 and NA22 two-particle correlations using a Lévy stable law

Parameter	UA1		NA22	
	Value	Error	Value	Error
$\mathcal{N}$	1.34	$\pm$ 0.03	0.95 (fixed)	
λ	1.85	$\pm$ 0.07	1.15	$\pm$ 0.17
R [fm]	6.36	$\pm$ 0.33	1.33	$\pm$ 0.30
α	0.49	$\pm$ 0.01	0.67	$\pm$ 0.07
χ^2/NDF	$56.5/42 = 1.35$		$27.67/35 = 0.769$	
CL	6.6 %		80.6%	

$\alpha_s(QCD) = \pi\alpha^2(BEC)/6$

UA1: 0.1 ~ 0.125 +- 0.005

NA22 0.14 ~ 0.23 +- 0.05

within errors even the running of $\alpha_s(QCD)$ is seen from BEC

Phases of QCD Matter

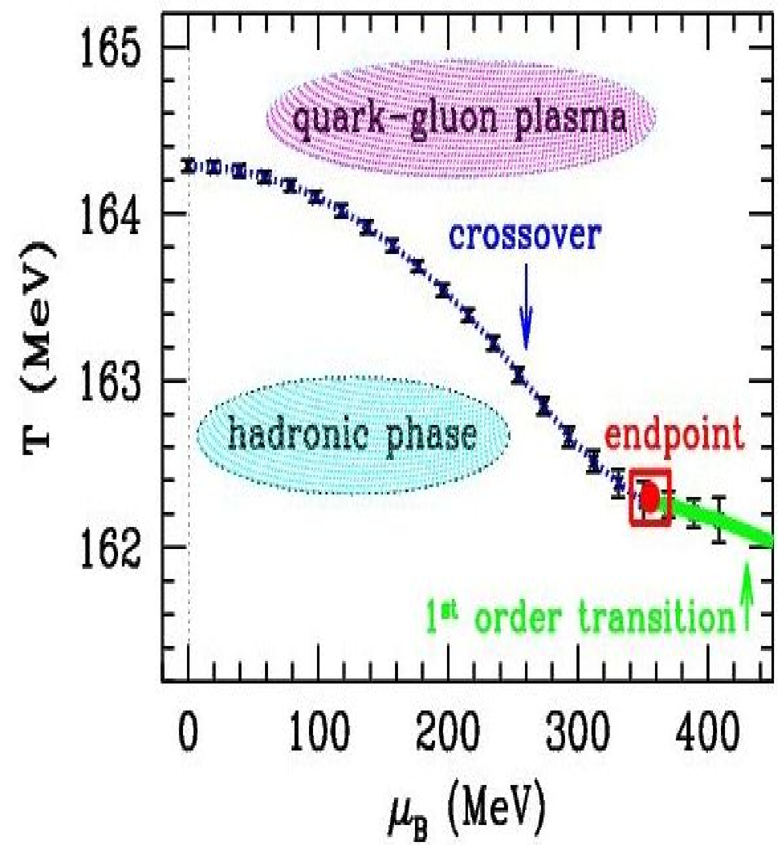
Quark Gluon Plasma

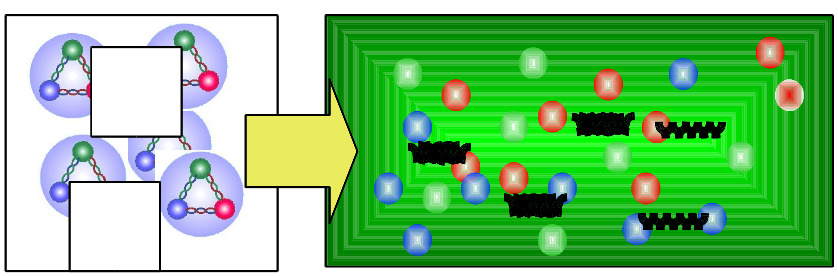
"Ionize" nucleons with heat

"Compress" them with density

New state(s?) of matter

Z. Fodor and S.D. Katz:
T_c = 164 ± 2 -> 190 MeV (QM'05)
even at finite baryon density,
Cross over like transition.
(hep-lat/0106002, hep-lat/0402006)





At the critical endpoint of the 1st order phase transition, the QCD phase transition is of second order.

Critical phenomena and Lévy Sources

At the critical point, in a second order phase transition

$$\rho(R) \sim R^{-(d-2+\eta)}$$

where η is the exponent of the correlation function

For Lévy stable sources,

$$\rho(R) \sim R^{-(1+\alpha)} \ ,$$

where α is the Lévy index of stability

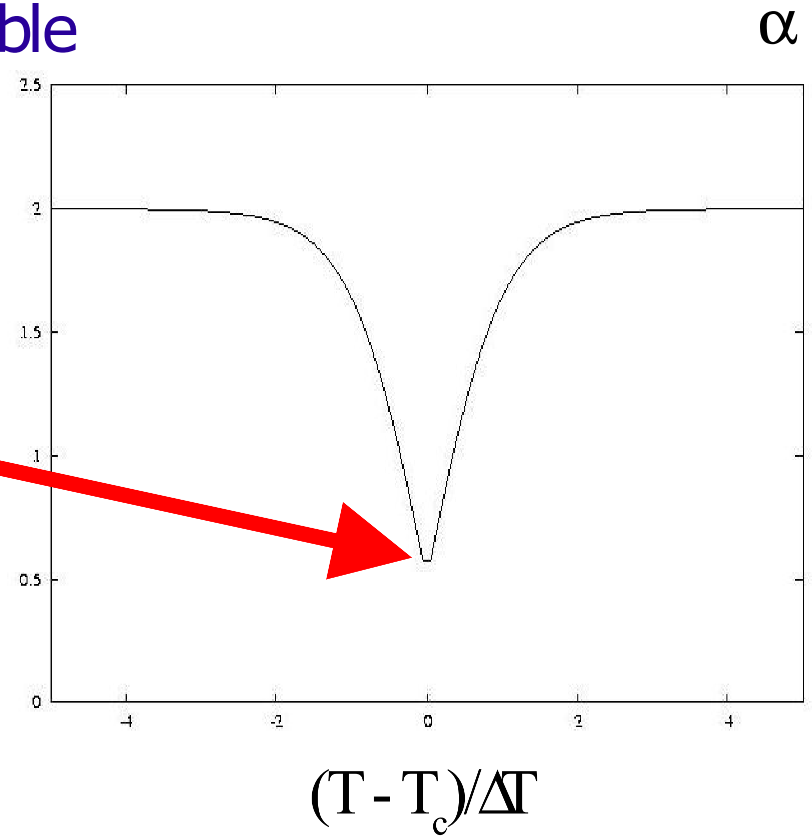
Critical phenomena and Lévy Sources

as the universality class of QCD
is that of the 3d Ising model,
(Stephanov-Rajagopal-Shuryak,hep-ph/9806219)

d=3 and the femtoscopy measurable
Lévy index of stability
for 2nd order QCD transitions
near the critical end point is:

$\alpha(\text{Lévy}) = \eta(\text{3d Ising}) \sim 0.50 +\!- \, 0.05$

Rieger, Phys. Rev. B 52, 6690 (1995)



Femptoscopy signals of various QGPs

strong 1st order
2nd order
cross-over
supercooled QGP (scQGP)

Strong 1st order QCD phase transition

(Pratt, Bertsch, Rischke, Gyulassy) Rout >> Rside

2nd order QGP -> hadron (Critical End Point)

(T. Cs, S. Hegyi, T. Novák, W.A. Zajc) α(Lévy) decreases to 0.5 near to critical sqrt(sNN)

cross-over QGP -> hadrons (Lattice QCD, Buda-Lund)

hadrons appear from a region of $T_0 > T_c$

supercooledQGP:a) hadron flash Rout~Rside~Rlong b) no cross-section effects

c) strangeness enhancement d) no ϕ -mass shift

scQGP = supercooled QGP predicted in (1994) is
not inconsistent with Au+Au data@RHIC in 2005

Summary and outlook

check the existence of the Lévy exponent in collisions
in p+p, d+Au and Au+Au @ RHIC,

Insert an extra parameter : Index of stability
when Gaussians start to fail
when Gaussians works seemingly well

relate a to the properties of QCD in p+p reactions:
Levy index of stability <-> anomalous dimension of QCD

Prediction: running of α(BEC) as given by the well-known α(QCD)

Interpretation in soft Au+Au: different domain,
for far from second order phase transition,

thermal source: $\alpha=2$

But near to the critical end point: related to
critical correlation exponents of 3d Ising model:

for a 2nd order QCD phase transition at the critical end point $\alpha\sim 0.5$.
A big change in the SHAPE of the correlation function at the critical $s_{NN}^{1/2}$.

R. Tagore: Playthings

Child, how happy you are sitting in the dust,
    playing with a broken twig all the morning.
I smile at your play
    with that little bit of a broken twig.
I am busy with my accounts,
    adding up figures by the hour.
Perhaps you glance at me and think,
    "What a stupid game to spoil your morning with!"
Child, I have forgotten
    the art of being absorbed in sticks and mud-pies.
I seek out costly playthings,
    and gather lumps of gold and silver.
With whatever you find you create your glad games,
    I spend both my time and my strength over things I never can obtain.
In my frail canoe I struggle to cross the sea of desire,
    and forget that I too am playing a game.